

# Volume-preserving actions of simple algebraic $\mathbb{Q}$ -groups on low-dimensional manifolds

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We prove that  $\mathrm{SL}(n, \mathbb{Q})$  has no nontrivial,  $C^\infty$ , volume-preserving action on any compact manifold of dimension strictly less than  $n$ . More generally, suppose  $\mathbf{G}$  is a connected, isotropic, almost-simple algebraic group over  $\mathbb{Q}$ , such that the simple factors of every localization of  $\mathbf{G}$  have rank  $\geq 2$ . If there does not exist a nontrivial homomorphism from  $\mathbf{G}(\mathbb{R})^\circ$  to  $\mathrm{GL}(d, \mathbb{C})$ , then every  $C^\infty$ , volume-preserving action of  $\mathbf{G}(\mathbb{Q})$  on any compact  $d$ -dimensional manifold must factor through a finite group.

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## 1. Introduction

The second author has conjectured that if  $\mathbf{G}$  is a simple algebraic  $\mathbb{Q}$ -group, and  $\mathrm{rank}_{\mathbb{R}} \mathbf{G} \geq 2$ , then every  $C^\infty$ , volume-preserving action of the arithmetic group  $\mathbf{G}(\mathbb{Z})$  on a compact manifold of small dimension must be *finite*. (This means that the action factors through the action of a finite group. See [2] for a precise statement of the conjecture and a survey of progress on this problem.) In this paper, we show that known results imply the analogue of the conjecture with  $\mathbf{G}(\mathbb{Q})$  in the place of  $\mathbf{G}(\mathbb{Z})$ . For example, we establish:

**Theorem 1.1.**  *$\mathrm{SL}(n, \mathbb{Q})$  has no nontrivial,  $C^\infty$ , volume-preserving action on any compact manifold of dimension strictly less than  $n$ .*

**Remark 1.2.**  $\mathrm{SL}(n, \mathbb{Q})$  contains large finite subgroups whenever  $n$  is large (such as an elementary abelian group of order  $2^{n-1}$ ). Therefore, topological arguments imply that if  $M$  is any compact manifold, then there is some  $n$ , such that  $\mathrm{SL}(n, \mathbb{Q})$  has no nontrivial,  $C^0$  action on  $M$  (see [6, Thm. 2.5]). However, unlike in Theorem 1.1, the

value of  $n$  depends on details of the topology of  $M$ , not just its dimension, because every finite group acts freely on some compact, connected, 2-dimensional manifold [1, Thm. 7.12].

The nontrivial part of Theorem 1.1 (namely, when  $n \geq 3$ ) is a special case of the following much more general result:

**Theorem 1.3.** *Assume:*

- (a)  $\mathbf{G}$  is an isotropic, almost-simple, linear algebraic group over  $\mathbb{Q}$ , such that, for every place  $v$  of  $\mathbb{Q}$ , the  $\mathbb{Q}_v$ -rank of every simple factor of  $\mathbf{G}(\mathbb{Q}_v)$  is at least two,
- (b)  $d \in \mathbb{Z}^+$ , such that there are no nontrivial, continuous homomorphisms from  $\mathbf{G}(\mathbb{R})^\circ$  to  $\mathrm{GL}(d, \mathbb{C})$ , and
- (c)  $G$  is a subgroup of finite index in  $\mathbf{G}(\mathbb{Q})$ .

*Then every  $C^\infty$ , volume-preserving action of  $G$  on any  $d$ -dimensional compact manifold  $M$  is finite.*

**Remark 1.4** (anisotropic groups). Assume, for simplicity, that  $\mathbf{G}$  is connected. Then the assumption that  $\mathbf{G}$  is isotropic can be eliminated if we add two hypotheses on the universal cover  $\tilde{\mathbf{G}}$ :

- (i)  $\tilde{\mathbf{G}}(\mathbb{Q})$  is projectively simple, and
- (ii) sufficiently large  $S$ -arithmetic subgroups of  $\tilde{\mathbf{G}}$  have the Congruence Subgroup Property.

Both of these hypotheses are known to be true unless  $\mathbf{G}$  is anisotropic of type  $A_n$ ,  $D_4$ , or  $E_6$ . See Remark 2.8 for more details.

- Remarks 1.5.** (1) To satisfy the requirement that the  $\mathbb{Q}_v$ -rank of every simple factor of  $\mathbf{G}(\mathbb{Q}_v)$  is at least two, it suffices to let  $\mathbf{G}$  be an absolutely almost-simple algebraic group over  $\mathbb{Q}$ , such that  $\mathrm{rank}_{\mathbb{Q}} \mathbf{G} \geq 2$ . In particular, we can take  $\mathbf{G} = \mathbf{SL}_n$  with  $n \geq 3$ . This yields Theorem 1.1.
- (2) The assumption that the subgroup  $G$  has finite index can be replaced with the weaker assumption that it contains the commutator subgroup  $[\mathbf{G}^\circ(\mathbb{Q}), \mathbf{G}^\circ(\mathbb{Q})]$ .
- (3) Our bound on the dimension  $d$  of  $M$  is probably not sharp. In particular, we conjecture that  $\mathrm{SL}(n, \mathbb{Q})$  has no volume-preserving action on any compact manifold of dimension strictly less than  $n^2 - 1$ . In the general case, it should suffice to assume that  $\mathbf{G}(\mathbb{R})^\circ$  has no simple factor of dimension  $\leq d$ .

## 2. Proof of Theorem 1.3

Assume the situation of Theorem 1.3. By passing to a subgroup of finite index, we assume that  $\mathbf{G}$  is connected.

- Notation 2.1.** (1)  $\tilde{\mathbf{G}}$  is the universal cover of  $\mathbf{G}$ . (We may realize  $\tilde{\mathbf{G}}$  as a Zariski-closed subgroup of  $\mathbf{SL}_N$ , for some  $N$  [5, Thm. 8.6, p. 63], so  $\tilde{\mathbf{G}}(R)$  is defined for any integral domain  $R$  of characteristic zero.)
- (2)  $\pi: \tilde{\mathbf{G}} \rightarrow \mathbf{G}$  is the natural homomorphism.
- (3)  $\mathbf{Z}$  is the kernel of  $\pi$  (so  $\mathbf{Z}$  is a finite, central  $\mathbb{Q}$ -subgroup of  $\tilde{\mathbf{G}}$ ).
- (4) If  $S$  is any finite set of prime numbers:
- (a)  $\mathbb{Z}_S$  is the ring of  $S$ -integers. That is,  $\mathbb{Z}_S = \mathbb{Z}[1/p_1, \dots, 1/p_r]$ , where  $S = \{p_1, \dots, p_r\}$ .
  - (b)  $\Gamma_S = \tilde{\mathbf{G}}(\mathbb{Z}_S)$ , so  $\Gamma_S$  is an  $S$ -arithmetic subgroup of  $\tilde{\mathbf{G}}$ .
  - (c)  $\hat{\Gamma}_S$  is the profinite completion of  $\Gamma_S$ .
- (5)  $\mathbb{Z}_p$  is the ring of  $p$ -adic integers, for any prime  $p$ .

We begin by recalling a few well-known facts about  $\tilde{\mathbf{G}}(\mathbb{Q})$ :

- Lemma 2.2.** (1) *Every proper, normal subgroup of  $\tilde{\mathbf{G}}(\mathbb{Q})$  is contained in the center of  $\tilde{\mathbf{G}}$ , and is therefore finite.*
- (2)  $\pi(\tilde{\mathbf{G}}(\mathbb{Q})) \subseteq G$ .
- (3)  $\mathbf{G}(\mathbb{Q})/\pi(\tilde{\mathbf{G}}(\mathbb{Q}))$  is an abelian group whose exponent divides  $|\mathbf{Z}(\mathbb{C})|$ . (In particular,  $\pi(\tilde{\mathbf{G}}(\mathbb{Q}))$  is a normal subgroup of  $\mathbf{G}(\mathbb{Q})$ .)

*Proof.* (1) See [3, Thm. 8.1]. This relies on our assumption that  $\mathbf{G}$  is isotropic.

(2) Since  $G$  has finite index in  $\mathbf{G}(\mathbb{Q})$ , it contains a finite-index subgroup of  $\pi(\tilde{\mathbf{G}}(\mathbb{Q}))$ . However, we know from (1) that  $\tilde{\mathbf{G}}(\mathbb{Q})$  has no proper subgroups of finite index. Therefore  $G$  must contain all of  $\pi(\tilde{\mathbf{G}}(\mathbb{Q}))$ .

(3) We have the following long exact sequence of Galois cohomology groups [8, (1.11), p. 22]:

$$H^0(\mathbb{Q}; \tilde{\mathbf{G}}) \longrightarrow H^0(\mathbb{Q}; \mathbf{G}) \longrightarrow H^1(\mathbb{Q}; \mathbf{Z}).$$

In other words,

$$\tilde{\mathbf{G}}(\mathbb{Q}) \longrightarrow \mathbf{G}(\mathbb{Q}) \xrightarrow{\delta} H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbf{Z}(\mathbb{C})).$$

Since  $\mathbf{Z}$  is central in  $\mathbf{G}$ , it is easy to see that the connecting map  $\delta$  is a group homomorphism. Therefore, the desired conclusion follows from the observation that multiplication by  $|\mathbf{Z}(\mathbb{C})|$  annihilates the abelian group  $H^1(*; \mathbf{Z}(\mathbb{C}))$ .  $\square$

Since  $\Gamma_S \subset \tilde{\mathbf{G}}(\mathbb{Q})$ , and  $G$  acts on  $M$ , Lemma 2.2(2) provides an action of  $\Gamma_S$  on  $M$  (for any  $S$ ). The following theorem about this action requires our assumption that there are no nontrivial, continuous homomorphisms from  $\mathbf{G}(\mathbb{R})^\circ$  to  $\text{GL}(\dim M, \mathbb{C})$ . It also uses our assumption that simple factors of  $\mathbf{G}(\mathbb{Q}_v)$  have rank at least two. (This implies that  $\mathbf{G}(\mathbb{Q}_v)$  has Kazhdan's property  $(T)$ .)

**Theorem 2.3** ([13, Cor. 1.3]). *If  $S$  is any finite set of prime numbers, then there exist*

- *a continuous action of a compact group  $K_S$  on a compact metric space  $X_S$ , and*

- a homomorphism  $\varphi_S: \Gamma_S \rightarrow K_S$ ,

such that the resulting action of  $\Gamma_S$  on  $X_S$  is measurably isomorphic (a.e.) to the action of  $\Gamma_S$  on  $M$ .

We may assume that  $\varphi_S(\Gamma_S)$  is dense in  $K_S$ . This implies:

**Lemma 2.4** (cf. [13, Cor. 1.5]).  *$K_S$  is profinite.*

*Proof.* It is an easy consequence of the Peter-Weyl Theorem that every compact group is a projective limit of compact Lie groups [4, Cor. 2.43, p. 51]. However, since  $\mathbf{G}(\mathbb{R})$  has no compact factors, the Margulis Superrigidity Theorem [7, Thm. B(iii), pp. 258–259] tells us that any homomorphism from  $\Gamma_S$  into a compact Lie group must have finite image. Since  $\varphi_S(\Gamma_S)$  is dense in  $K_S$ , this implies that  $K_S$  is a projective limit of finite groups, as desired.  $\square$

Therefore, we may assume  $K_S$  is the profinite completion  $\widehat{\Gamma}_S$  of  $\Gamma_S$ . We have the following well-known description of  $\widehat{\Gamma}_S$  (because  $\mathbf{G}$  is isotropic).

**Theorem 2.5** (Congruence Subgroup Property [9,10,11]). *If  $S$  is nonempty, then the natural inclusion  $\Gamma_S \hookrightarrow \times_{p \notin S} \widetilde{\mathbf{G}}(\mathbb{Z}_p)$  extends to an isomorphism  $\widehat{\Gamma}_S \cong \times_{p \notin S} \widetilde{\mathbf{G}}(\mathbb{Z}_p)$ .*

Fix a prime number  $q \neq 2$ . The inclusion  $\Gamma_{\{2\}} \subset \Gamma_{\{2,q\}}$  provides us with an action of  $\Gamma_{\{2\}}$  on  $X_{\{2,q\}}$ , but this must be isomorphic to the action of  $\Gamma_{\{2\}}$  on  $X_{\{2\}}$  (since both are isomorphic to the action on  $M$ ). Therefore, the action of  $\widehat{\Gamma}_{\{2\}}$  on  $X_{\{2\}}$  must factor through  $\widehat{\Gamma}_{\{2,q\}}$  (a.e.). Furthermore, if we use the Congruence Subgroup Property (2.5) to identify  $\widehat{\Gamma}_S$  with  $\times_{p \notin S} \widetilde{\mathbf{G}}(\mathbb{Z}_p)$ , then it is obvious that  $\widetilde{\mathbf{G}}(\mathbb{Z}_q)$  is in the kernel of the homomorphism  $\widehat{\Gamma}_{\{2\}} \rightarrow \widehat{\Gamma}_{\{2,q\}}$ . Therefore,  $\widetilde{\mathbf{G}}(\mathbb{Z}_q)$  acts trivially on  $X_{\{2\}}$  (a.e.).

Since the subgroups  $\widetilde{\mathbf{G}}(\mathbb{Z}_q)$  generate a dense subgroup of  $\times_{p \neq 2} \widetilde{\mathbf{G}}(\mathbb{Z}_p) \cong \widehat{\Gamma}_{\{2\}}$ , we conclude that  $\widehat{\Gamma}_{\{2\}}$  acts trivially (a.e.). Therefore,  $\Gamma_{\{2\}}$  acts trivially on  $M$  (not just a.e., because  $\Gamma_{\{2\}}$  acts continuously on  $M$ ), so the action of  $\widetilde{\mathbf{G}}(\mathbb{Q})$  has an infinite kernel. Hence, Lemma 2.2(1) implies that the kernel is all of  $\widetilde{\mathbf{G}}(\mathbb{Q})$ . This means that  $\widetilde{\mathbf{G}}(\mathbb{Q})$  acts trivially on  $M$ .

So the action of  $G$  factors through  $G/\pi(\widetilde{\mathbf{G}}(\mathbb{Q}))$ . From Lemma 2.2(3), we know that this quotient is an abelian group of finite exponent, so the corollary of the following theorem tells us that the action is finite.

**Theorem 2.6** ([6, Thm. 2.5]). *If  $A$  is any abelian group of prime exponent, then every  $C^0$  action of  $A$  on any compact manifold is finite.*

**Corollary 2.7.** *If  $A$  is any abelian group of finite exponent, then every  $C^0$  action of  $A$  on any compact manifold is finite.*

*Proof.* We can assume the exponent of  $A$  is a power of a prime  $p$  (because  $A$  is the direct product of its finitely many Sylow subgroups). We can also assume that

the action of  $A$  is faithful, so the theorem tells us that  $A$  has only finitely many elements of order  $p$ . This means the kernel of the homomorphism  $x \mapsto x^p$  is finite, so it is easy to prove by induction that  $A$  has only finitely many elements of any order  $p^k$ . Since  $A$  has finite exponent, this implies that  $A$  is finite.  $\square$

This completes the proof of Theorem 1.3. We now discuss the generalization described in Remark 1.4.

**Remark 2.8.** The assumption that  $\mathbf{G}$  is isotropic was used in only two places: the projective simplicity of  $\tilde{\mathbf{G}}(\mathbb{Q})$  (Lemma 2.2(1)) and the Congruence Subgroup Property (2.5).

The projective simplicity is known to be true unless  $\mathbf{G}$  is anisotropic of type  $A_n$ ,  ${}^3,6D_4$ , or  $E_6$  [8, pp. 513–515]. (Projective simplicity obviously fails if there is a nonarchimedean place  $v$ , such that  $\mathbf{G}(\mathbb{Q}_v)$  has a compact factor [8, pp. 510–511]. However, compact nonarchimedean factors cannot arise unless  $\mathbf{G}$  is of type  $A_n$  [8, Thm. 6.5, p. 285]. In any case, we have ruled out compact factors by requiring the simple factors of  $\mathbf{G}(\mathbb{Q}_v)$  to have rank at least 2.) When there are no compact factors, projective simplicity is also known to be true for inner forms of type  ${}^1A_n$  [12, p. 180].

For the Congruence Subgroup Property, it suffices to assume that every prime number  $q$  is contained in a finite set  $S$  of prime numbers, such that the congruence kernel  $C(S, \tilde{\mathbf{G}})$  is central. (This condition is known to be true unless  $\mathbf{G}$  is anisotropic of type  $A_n$ ,  $D_4$ , or  $E_6$  [8, Thms. 9.23 and 9.24, pp. 568–569]. In fact, for our purposes, it would suffice to know that  $C(S, \tilde{\mathbf{G}})$  is abelian.) To see that this assumption suffices, note that, for any finite set  $S$  of prime numbers, Strong Approximation [8, Thm. 7.12, p. 427] tells us  $\hat{\Gamma}_S/C(S, \tilde{\mathbf{G}}) \cong \times_{p \notin S} \tilde{\mathbf{G}}(\mathbb{Z}_p)$ . In particular,  $\hat{\Gamma}_\emptyset/C(\emptyset, \tilde{\mathbf{G}}) \cong \times_p \tilde{\mathbf{G}}(\mathbb{Z}_p)$ , so, for each prime  $q$ , we may let  $\hat{\mathbf{G}}(\mathbb{Z}_q)$  be the inverse image of  $\tilde{\mathbf{G}}(\mathbb{Z}_q)$  in  $\hat{\Gamma}_\emptyset$ . The homomorphism  $\hat{\Gamma}_\emptyset \rightarrow \hat{\Gamma}_S$  must map  $\hat{\mathbf{G}}(\mathbb{Z}_q)$  into  $C(S, \tilde{\mathbf{G}})$  for all  $q \in S$ . If  $C(S, \tilde{\mathbf{G}})$  is abelian, this implies that the image of the commutator subgroup  $[\hat{\mathbf{G}}(\mathbb{Z}_q), \hat{\mathbf{G}}(\mathbb{Z}_q)]$  is trivial. Since this is true for all  $q$ , we conclude that  $[\Gamma_\emptyset, \Gamma_\emptyset]$  acts trivially on  $M$ . This is sufficient to show that  $\tilde{\mathbf{G}}(\mathbb{Q})$  acts trivially.

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